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Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control*

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1. INTRODUCTION

The problem of intransitivities in social choice has been the subject of much investigation since Arrow's pioneering work in this area. In the context of social choice over multidimensional policy spaces, Plott [10] has shown the severity of the restrictions which are needed in order to generate an equilibrium policy outcome. Little attention has been paid, however, to the properties of the intransitivities when these strong equilibrium conditions are not met. One exception is Tullock [13], who has argued that Arrow's result is irrelevant in this context because the cycle set will be a fairly small area in the space. But Tullock's argument is not rigorous, and no other work has proceeded any further along this line.

In this paper, we show a rather surprising result, namely, that in the case where all voters evaluate policy in terms of a Euclidian metric, if there is no equilibrium outcome, then the intransitivities extend to the whole policy space in such a way that all points are in the same cycle set. The implications of this result are that it is theoretically possible to design voting procedures which, starting from any given point, will end up at any other point in the space of alternatives, even at Pareto dominated ones. A constructive proof is given below which does precisely this in the Euclidian case. While we only consider the case of Euclidian metrics here, there does not seem to be any reason why the results herein would not extend to more general types of utility functions.

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2. Assumptions and Definitions

We assume a set $N = \{1, 2,..., n\}$ of *voters*, and assume that the *policy space* X is Euclidian m space, i.e., $X = R^m$. For each voter $i \in N$, we assume there is a utility function $U_i : X \to R$ which for present purposes is assumed to be a monotone decreasing function of Euclidian distaince; i.e., for all $i \in N$, $\exists x_i \in R^m$ s.t.

$$U_i(x) = \Phi_i \| x - x_i \|. \tag{2.1}$$

Here $\| \cdot \|$ represents the standard Euclidian norm, and $\Phi_i : R \to R$ is any strictly monotone decreasing function. We use the notation

$$\begin{aligned} x \succ_i y &\Leftrightarrow U_i(x) > U_i(y), \\ x \succcurlyeq_i y &\Leftrightarrow U_i(x) \geqslant U_i(y). \end{aligned} \tag{2.2}$$

Given the nature of the utility functions it follows that

$$x \succ_i y \Leftrightarrow \| x - x_i \| < \| y - x_i \|. \tag{2.3}$$

We use the notation $| B |$ to represent the number of elements in a set $B \subseteq N$, and use the shorthand $| x \succcurlyeq_i y | = |\{i \in N \mid x \succcurlyeq_i y\}|$. Then, we can define a majority preference relation over R^m as follows. For any x, $y \in R^m$

$$x \succcurlyeq y \Leftrightarrow | x \succcurlyeq_i y | \geqslant n/2. \tag{2.4}$$

Defining the strong majority relation in the usual way (i.e., $x \succ y \Leftrightarrow x \succcurlyeq y$ and $\sim(y \succcurlyeq x)$), it follows that

$$x \succ y \Leftrightarrow | x \succ_i y | > n/2. \tag{2.5}$$

If all voters evaluate policy in terms of Euclidian distance, the conditions for equilibria can be stated in terms of the existence of a total median. We develop this formally:

For any $y \in R^m$, $c \in R$ we can define a hyperplane as follows:

$$H_{y,c} = \{x \mid x' \cdot y = c\}. \tag{2.6}$$

This partitions R^n into three sets, $H_{y,c}$, $H_{y,c}^+$, and $H_{y,c}^-$, where

$$\begin{aligned} H_{y,c}^+ &= \{x \mid x' \cdot y > c\}, \\ H_{y,c}^- &= \{x \mid x' \cdot y < c\}. \end{aligned} \tag{2.7}$$

Now, for any $S \subseteq R^m$, we write $| S | = |\{i \mid x_i \in S\}|$. Then $H_{y,c}$ is said to be a *median hyperplane* $\Leftrightarrow |H^+_{y,c}| \leqslant n/2$ and $|H^-_{y,c}| \leqslant n/2$. We let $\mathbf{M}$ be be the set of median hyperplanes. It is proved in [6] that for all $y \in R^m$, there is at least one $H_{y,c} \in \mathbf{M}$, although this may not be unique.

DEFINITION 1. *A vector $x^* \in X$ is a total median iff for all $y \in R^m$, $\exists H_{y,c} \in \mathbf{M}$ such that $x^* \in H_{y,c}$. It is a strong total median if in addition, for all y, $H_{y,c} \in \mathbf{M}$ is unique.*

A total median is not necessarily unique, but a strong total median is unique. Notice that whenever there are an odd number of voters, any total median is unique, and is also strong. For even numbers of voters it is possible to have a unique total median which is not strong, as would be the case if four voters were arranged with their ideal points at the corners of a square.

DEFINITION 2. *A vector $x^* \in X$ is a majority Condorcet point iff $x^* \succcurlyeq y$ for all $y \in X$.*

Davis, Degroot, and Hinich [1] prove the following theorem, which establishes necessary and sufficient conditions for the existence of a majority Condorcet point and for transitive social ordering in the Euclidian model.

THEOREM 1. *If all U_i are as in (2.1), then $x^* \in X$ is a Condorcet point iff it is a total median. Further, if x^* is a strong total median, the social order is transitive on X, with $x \succcurlyeq y \Leftrightarrow \| x - x^* \| \leqslant \| y - x^* \|$.*

Proof. See [1, Theorems 1 and 4, and Corollary 2]. Q.E.D.

Figure 1 illustrates the necessity of the strong total median to guarantee transitivity of the social ordering. Here there is a unique total median at $x^* = \sum_{i=1}^{4} x_i/4$, but it is not a strong total median. In this example, we have $z \sim x$, $x \sim y$, yet $y \succ z$, violating transitivity of the social ordering.

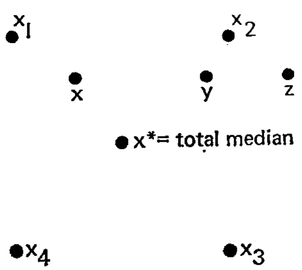

FIGURE 1

With the exception of the above type of problem, generated by even numbers of voters, which gives rise to intransitive indifference, Theorem 1 shows that the existence of Condorcet points and the existence of transitivity of the social ordering both coincide. For odd numbers of voters, the two properties completely coincide. This result is not too surprising, but given the severity of the restrictions needed to guarantee transitivity (namely, existence of a strong total median) it is of considerable interest to explore the nature of the intransitivities when these symmetry conditions are not met.

3. The Extent of Intransitivities

In this section, we show that when transitivity breaks down, it *completely* breaks down, engulfing the whole space in a single cycle set. The slightest deviation from the conditions for a Condorcet point (for example, a slight movement of *one* voter's ideal point) brings about this possibility:

THEOREM 2. *Assume $m \geqslant 2$, $n \geqslant 3$, and all voters have utility functions as in* (2.1). *If there is no total median, then for any x, $y \in X$, it is possible to find a sequence of alternatives, $\{\theta_0, \ldots, \theta_N\}$ with $\theta_0 = x$, $\theta_N = y$, such that $\theta_{i+1} \succ \theta_i$ for $0 \leqslant i \leqslant N-1$.*

Proof. For each $y \in R^m$, with $\| y \| = 1$, define $C_y \subseteq R$ to be the set of c satisfying $\{x \mid x \cdot y = c\} \in \mathbf{M}$. It is easily shown that C_y is a closed interval. So, setting $c_y = \inf C_y$, it follows that $c_y \in C_y$, and hence, we define H_y, for any y, as

$$H_y = \{x \mid x' \cdot y = c_y\} \in \mathbf{M}. \tag{3.1}$$

Now it can be shown that a total median exists iff there is an $x^* \in R^m$ with

$$x^* \in \bigcap_{\|y\|=1} \bar{H}_y^+, \tag{3.2}$$

where $\bar{H}_y^+ = H_y \cup H_y^+ = \{x \mid x' \cdot y \geqslant c_y\}$.

Since there is no total median, it follows that there is no common solution to the above system of inequalities. By Helley's theorem, it follows that we can find a set of $m+1$ vectors, $y_0, \ldots, y_m$, with no common solution to

$$x^{*\prime} \cdot y_i \geqslant c_{y_i} = c_i . \tag{3.3}$$

Out of this set, we pick a subset of vectors with no common solution (without loss of generality assume they are the first $p + 1$ vectors, $y_0, \ldots, y_p$), such that for any j, $0 \leqslant j \leqslant p$, there *is* a common solution to

$$x^{*\prime} \cdot y_i \geqslant c_i \qquad \text{for} \quad i \neq j,\ 0 \leqslant i \leqslant p. \tag{3.4}$$

For each $0 \leqslant j \leqslant p$ we set z_j to be a solution to

$$z_j' \cdot y_i = c_i \qquad \text{for all } i \neq j,\ 0 \leqslant i \leqslant p, \tag{3.5}$$

and set $z = (1/(p+1)) \sum_{j=0}^{p} z_j$; we assume without loss of generality that the origin of the vector space is at z (i.e., $z = 0$, the 0 vector). Then it follows that $c_i > 0$, for all $0 \leqslant i \leqslant p$, because

$$0 = z' \cdot y_i = \sum_{j=0}^{p} z_j' \cdot y_i = pc_i + z_i' \cdot y_i < (p+1)\, c_i\,. \tag{3.6}$$

Further, for any $x \in R^m$, note that

$$x' \cdot y_i \leqslant 0 \qquad \text{for some } 0 \leqslant i \leqslant p. \tag{3.7}$$

Otherwise for some large $\alpha \in R$, αx is a common solution for $\alpha x' \cdot y_i \geqslant c_i$ for $0 \leqslant i \leqslant p$, a contradiction. Setting $H_i = H_{y_i}$, Fig. 2 illustrates a possible configuration of the y_i and H_i for the two-dimensional case.

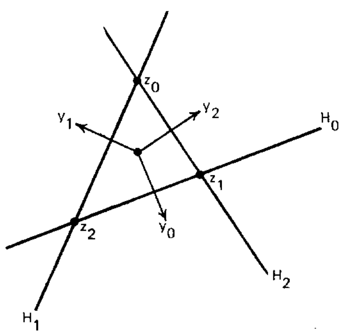

FIGURE 2

Now, for any θ_k, we construct θ_{k+1} as follows: From (3.7), it follows that for some i, $\theta_k' \cdot y_i \leqslant 0$. Pick any such i. Then, we define θ_{k+1} as follows:

$$\theta_{k+1} = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i\,. \tag{3.8}$$

Figure 3 illustrates this for the two-dimensional case. Now,

$$\begin{aligned} \| \theta_k \|^2 &= \|(y_i' \cdot \theta_k)\, y_i + (\theta_k - (y_i' \cdot \theta_k)\, y_i)\|^2 \\ &= \|(y_i' \cdot \theta_k)\, y_i \|^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2 \\ &= (y_i' \cdot \theta_k)^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2, \end{aligned} \tag{3.9}$$

and similarly

$$\| \theta_{k+1} \|^2 = (y_i' \cdot \theta_{k+1})^2 + \| \theta_{k+1} - (y_i' \cdot \theta_{k+1})\, y_i \|^2, \tag{3.10}$$

but, from (3.8),

$$\begin{aligned} &\theta_{k+1} - (y_i' \cdot \theta_{k+1})\, y_i \\ &\quad = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i - y_i' \cdot (\theta_k + (c_i - 2y_i' \cdot \theta_k)\, y_i)\, y_i \\ &\quad = \theta_k + [c_i - 2y_i' \cdot \theta_k]\, y_i - [c_i - 2y_i' \cdot \theta_k]\, y_i - (y_i' \cdot \theta_k)\, y_i \\ &\quad = \theta_k - (y_i' \cdot \theta_k)\, y_i\,. \end{aligned} \tag{3.11}$$

So, substituting (3.11) in (3.10), we get

$$\begin{aligned} \| \theta_{k+1} \|^2 &= (y_i' \cdot \theta_{k+1})^2 + \| \theta_k - (y_i' \cdot \theta_k)\, y_i \|^2 \\ &= (y_i' \cdot \theta_{k+1})^2 - (y_i' \cdot \theta_k)^2 + \| \theta_k \|^2 \\ &= \| \theta_k \|^2 + \Delta, \end{aligned} \tag{3.12}$$

where $\Delta = (y_i' \cdot \theta_{k+1})^2 - (y_i' \cdot \theta_k)^2$. But, now, using (3.8),

$$\begin{aligned} \Delta &= [y_i' \cdot (\theta_k + (c_i - 2y_i' \cdot \theta_k)\, y_i)]^2 - (y_i' \cdot \theta_k)^2 \\ &= [y_i' \cdot \theta_k + (c_i - 2y_i' \cdot \theta_k)]^2 - (y_i' \cdot \theta_k)^2 \\ &= [c_i - y_i' \cdot \theta_k]^2 - (y_i' \cdot \theta_k)^2 \\ &= c_i^2 - 2c_i y_i' \cdot \theta_k \geqslant c_i^2 \end{aligned} \tag{3.13}$$

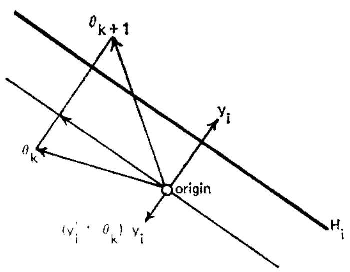

FIGURE 3

since $y_i' \cdot \theta_k \leqslant 0$ and $c_i > 0$. Hence,

$$\| \theta_{k+1} \|^2 \geqslant \| \theta_k \|^2 + c_i^2. \tag{3.14}$$

It is obvious, then, by successive application of the above algorithm, we can get θ_j as far from the origin as we want.

Next we prove that $\theta_{k+1} \succ \theta_k$. To see this, note that for any $j \in N$,

$$\begin{aligned} \theta_{k+1} \succ_j \theta_k &\Leftrightarrow \| x_j - \theta_{k+1} \| < \| x_j - \theta_k \| \\ &\Leftrightarrow x_j \cdot (\theta_{k+1} - \theta_k) > ((\theta_{k+1} + \theta_k)'/2)\,(\theta_{k+1} - \theta_k) \\ &\Leftrightarrow x_j' \cdot y_i > ((\theta_{k+1} + \theta_k)/2)' \cdot y_i \\ &\Leftrightarrow x_j' \cdot y_i > c_i/2. \end{aligned}$$

But now, since $H_i = \{x \mid x' \cdot y_i = c_i\} \in \mathbf{M}$ and, by assumption, $\{ x \mid x' \cdot y_i = c_i/2\} \notin \mathbf{M}$, it follows that $|\{x \mid x' \cdot y_i > c_i/2\}| > n/2$ hence, $| \theta_{k+1} \succcurlyeq_j \theta_k | > n/2$ and it follows that $\theta_{k+1} \succ \theta_k$, as we wanted to show.

Thus, we have a sequence $\{\theta_1, \theta_2, \ldots\}$ such that

$$\theta_{k+1} \succ \theta_k \tag{3.15}$$

and such that

$$\| \theta_k \| \to \infty \qquad \text{as} \quad k \to \infty. \tag{3.16}$$

But now, we must show that for any x, y, we can construct a sequence satisfying (3.15) and (3.16), such that $\theta_0 = x$, and $\theta_N = y$. There is no problem with θ_0, but we must show we can get $\theta_N = y$.

To show this, we simply take $B = \{x \mid \| x \| < \rho\}$ to be a sphere of radius ρ satisfying $| B | > n/2$ and $y \in B$. Then, we set $B^* = \{x \mid \| x \| > 3\rho\}$.

It follows, for any $\theta \in B^*$, that $y \succ \theta$, since for any $x_i \in B$, $\| \theta - x_i \| > 2\rho$, and $\| y - x_i \| < 2\rho$.

Hence, we pick a sequence $\{\theta_0, \ldots, \theta_{N-1}\}$ satisfying (3.15) and (3.16) with $\theta_0 = x$, $\theta_{N-1} \in B^*$. Then we set $\theta_N = y$, and from the above argument, $\theta_N \succ \theta_{N-1}$. But then $\{\theta_0, \ldots, \theta_N\}$ is a sequence of proposals satisfying

$$\begin{aligned} &\theta_0 = x, \\ &\theta_N = y, \\ &\theta_{i+1} \succ \theta_i, \qquad 0 \leqslant i \leqslant N-1, \end{aligned} \tag{3.17}$$

and we are done. Q.E.D.

In Fig. 4, we illustrate the above algorithm for a simple example with five voters in two dimensions. Here we construct a cycle which arrives at a Pareto dominated point y, from a Pareto optimal point x. Note that the algorithm given is not necessarily the most efficient way of getting from x to y. In particular, as illustrated here, it is seldom necessary to actually

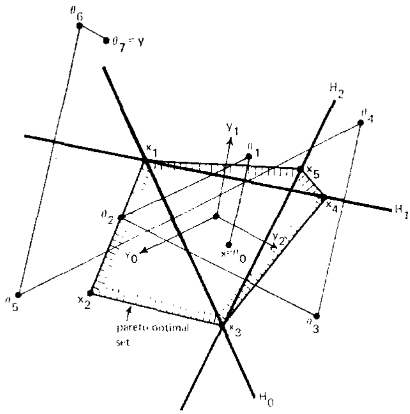

FIGURE 4

get θ_{N-1} in B^*. Frequently one will obtain a θ_k prior to this stage which will beat y.

4. CONCLUSION

The theorem of the previous section shows that, at least for the Euclidian case, either the majority rule social order is completely transitive, or it is involved in a single cycle set. This result is of course dependent on the assumption of Euclidian utility functions. It seems probable, however, that the results would extend to a much larger class of utility functions. In particular it is conjectured that the same type of result would hold if each utility function were separable, i.e., of the form

$$U_i(x) = \sum_{j=1}^{m} U_{ij}(x^j),$$

where x^j is the jth component of the vector x, and where U_{ij} is any real valued function.

In cases where majority rule is not transitive, attempts have been made in the literature to isolate subsets of alternatives which are either more stable or are in some sense normatively better than other points in the space. Some of these attempts have been based on various definitions of "top" cycle sets. Kadane [3] shows the vector of medians is always in such a set in a multidimensional model, and in a more general framework, the idea of top cycles serves as a basis for Schwartz's GOCHA set [11, 12] (called $O(a, s)$ in [11]). In the Euclidean example of this paper, the top cycle set includes the whole of R^m. If the results here extend to more general utility functions, it would suggest that such generalized equilibrium notions may not be too powerful in infinite alternative spaces.

The existence of a single cycle set implies that it is *possible* for majority rule to wander anywhere in the space of alternatives. The *likelihood* of this occurring probably is strongly dependent on the nature of the institutional mechanisms which generate the agenda. In the context of two-party competition, McKelvey and Ordeshook [6] prove, in the Euclidian case, that mixed strategy solutions are limited to the set of "partial medians," and recently Kramer [5] has shown that in a sequence of elections, where each candidate attempts to maximize plurality against the position of the previous winning candidate, that candidates converge towards the "minimax" set. Both the minimax set and the set of partial medians always exist, and tend to be small and centrally located subsets of the Pareto optimals. For the above institutional mechanisms then, the existence of a single cycle set would be largely irrelevant, and the conclusions of Tullock [13] basically confirmed.

When there is the possibility of *control* of the agenda, either exogenously or by some member of the voting body, the existence of a single cycle set *would* be of considerable importance, as can be illustrated for the Euclidian case. From [8, 10], it follows that the existence of a Condorcet point is equivalent to a type of weak symmetry between the voters. Weak symmetry occurs when it is possible to find a point, x^*, such that voters can be divided into pairs with ideal points in opposite directions from x^*. Thus, if voters i and j are paired, we must have $(x_i - x^*) = -\alpha(x_j - x^*)$ for some $\alpha > 0$, as in Fig. 5. Any remaining voter (at most one) must be at the point x^*. This condition of weak symmetry is equivalent to the existence of a total median at x^*. With an odd number of voters it is equivalent to existence of a strong total median. Given the severity of the above conditions, the chances are very slim that such a point will exist in any particular situation. Even if a strong total median exists, it is possible for any *one* voter, by changing his preferences, to eliminate this total median. Thus, in the illustration of Fig. 5, Voter 8 could misrepresent

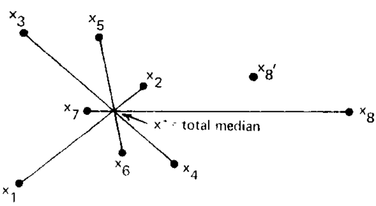

FIGURE 5

his preferences by voting *as if* he had a Euclidian utility function centered at x_8' instead of at x_8. Now there is no total median, and we are in a state of flux as described in Theorem 2.

It follows from the above consideration that if any one voter, say the "Chairman," has complete control over the agenda (in the sense that he can choose, at each stage of the voting, any proposal $\theta_i \in R^n$ to be considered next) that he can construct an agenda which will arrive at any point in space, in particular at his ideal point. Even if there is a majority rule equilibrium, as in Fig. 5, the Chairman (say Voter 8) could construct an agenda which terminates at his actual ideal point (x_8) by first misrepresenting his preferences to create the intransitivities and then applying Theorem 2 to choose the appropriate agenda. This type of manipulation is possible *regardless* of the preferences of the other voters and regardless of whether the "sincere" social ordering is transitive.

The possibility outlined here for controlling the social outcome through control of the agenda depends on several assumptions which are implicit in the above scenario but which should be made more explicit. First, the Chairman must have perfect information of the other voters' preferences in order to design such an agenda. In light of the above analysis, it would obviously not be in the other voters' interests to supply such information. Second, it depends on individuals being able to make fine distinctions between alternatives without becoming indifferent. The algorithm of Theorem 2 depends on finding new alternatives which some pivotal voters just barely prefer to the previous motion. If voters cannot make such fine distinctions, this could impose some limits on the space of intransitivities such that Theorem 2 would no longer hold. Finally, the result depends on other voters voting sincerely and without collusion. If the other voters see what is occurring and know what agenda is being used they might, even without collusion, vote against their preferences at some stage (i.e., vote sophisticatedly) in order to outwit our clever Chairman. Gibbard [2] and Pattanaik [9] show that such consideration cannot be ruled out in general and Kramer [4] analyzes such behavior in a multidimensional context, proving the existence of an equilibrium to the above model if sophistication is taken into account. If collusion occurs, then one must model the above as an n-person game without sidepayments (see [6]), and for all practical purposes, the chairman loses his power since any coalition can ensure any particular alternative in a given agenda by voting appropriately as a bloc at each stage of the agenda. Nevertheless, subject to the qualifications made above, the result of this paper, if it can be generalized, suggests that control of the agenda may be a powerful tool in a "naive" voting body.

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